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LETTER TO THE EDITOR

The universality of period-doubling bifurcations in certain two-dimensional reversible area-preserving mappings with quadratic nonlinearity

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Abstract. We show that all the period-doubling bifurcation sequences of two-dimensional one-parameter reversible area-preserving mappings of a certain form with quadratic nonlinearity are equivalent. Two such sequences are calculated according to the theory developed by the author for the explicit demonstration of the equivalence.

There have been several period-doubling bifurcation sequences, often called the Feigenbaum sequence after his discovery of such a sequence in 1D mappings (Feigenbaum 1978, 1979), for the 2D area-preserving one-parameter mapping with a quadratic nonlinearity, studied and reported (Bountis 1981, Greene *et al* 1981, Helleman 1980). The universality of such a bifurcation scheme is recognised and the universal constants are estimated from the sequences. However, it has not been recognised that these sequences are actually equivalent to each other, let alone universal.

In this letter we point out that all the one-parameter mappings with quadratic nonlinearity are equivalent to each other in the sense that there is one-to-one or one-to-two correspondence. This implies that all the Feigenbaum sequences of the one-parameter mappings with quadratic nonlinearity are equivalent. As an explicit demonstration, we present two independently calculated Feigenbaum sequences of quadratic mappings of different kinds according to the theory developed by the author (Lee and Choi 1983).

We used in the following the form given in the paper by the author (Lee and Choi 1983) for the 2D area-preserving mappings as

$$T = \begin{cases} x_{n+1} = -y_n + 2h(x_n) \\ y_{n+1} = x_n. \end{cases} \tag{1}$$

This mapping becomes Helleman's standard quadratic mapping (Helleman 1980) if we choose for $h(x)$

$$h_{\mathrm{HL}}(x) = cx + x^2, \tag{2}$$

De Vogelaere's quadratic mapping (Greene *et al* 1981) if we choose

$$h_{\mathrm{DV}}(x) = px - (1-p)x^2, \tag{3}$$

and Hénon's quadratic mapping (Bountis 1981) if we choose for $h(x)$

$$h_{\mathrm{HN}}(x) = \tfrac{1}{2}(1 - ax^2). \tag{4}$$

As pointed out by Helleman (1981), a quadratic nonlinearity is unchanged by a translation of the origin. If we choose the origin at the fixed point of period 1, we can write the $h(x)$ of equation (1) in the form

$$h(x) = px + q(p)x^2 \tag{5}$$

where $q(p)$ is some function of p. If we rescale x and y by

$$\xi = q(p)x \qquad \eta = q(p)y,$$

the mapping (1) can be brought into Helleman's standard form

$$\xi_{n+1} = -\eta_n + h(\xi_n) \qquad \eta_{n+1} = \xi_n,$$

with $h(x) = h_{\mathrm{HL}}(x)$ of equation (2).

Table 1. Feigenbaum sequence of period-doubling bifurcations of Hénon's quadratic mapping, (1) and (4) on the symmetry line $x = \frac{1}{2}(1 - ay^2)$ with accumulation point $a_c = 4.136166\ldots$.

(*a*) Parameter values at which the n-period orbit becomes unstable ($\phi'_n(y^*_n) = -2$), $\phi'_n(y^*_n)$, the initial points on the symmetry line.

k	$n = 2^k$	a_k	$\phi'_n(a_k, y^*_n)$	x^*_n	y^*_n
0	1	+3.000 000 000 000	−2.000 000 000 000	+0.333 333 333 333	+0.333 333 333 333
1	2	+4.000 000 000 000	−2.000 000 000 000	+0.500 000 000 000	+0.000 000 000 000
2	4	+4.120 452 497 319	−1.999 999 999 991	+0.492 637 586 768	+0.059 779 549 344
3	8	+4.134 363 912 468	−2.000 000 001 230	+0.495 893 250 224	+0.044 571 758 452
4	16	+4.135 960 058 811	−2.000 000 003 680	+0.495 180 967 141	+0.048 273 276 101
5	32	+4.136 143 097 399	−2.000 000 058 700	+0.495 364 528 249	+0.047 343 917 327
6	64	+4.136 164 085 605	−1.999 999 990 600	+0.495 319 292 500	+0.047 574 241 131
7	128	+4.136 166 492 190	−2.000 107 064 000	+0.495 330 584 451	+0.047 516 807 529
8	256	+4.136 166 768 178	−1.999 586 740 000	+0.495 327 776 999	+0.047 531 088 365
9	512	+4.136 166 799 826	−1.984 365 790 000	+0.495 328 472 923	+0.047 527 548 192

Table 1(*b*). Sequences leading scaling constants δ, α, β. $T^{n/4}$ iteration of the pair of the fixed points on the symmetry line is the pair bifurcated off the symmetry line from the $\frac{1}{2}n$ period orbit. They always bifurcate parallel to the x axis (Lee *et al* 1983). In the above

$$\delta_k = (a_{k-1} - a_k)/(a_k - a_{k+1}), \qquad \alpha_{k+1/2} = [y^*_n - y^{n/2}(x^*_n y^*_n)]/[y^*_{2n} - y^n(x^*_{2n} y^*_{2n})]$$

$$\beta_{k+1/2} = [x^{n/4}(x^*_n, y^*_n) - x^{3n/4}(x^*_n, y^*_n)]/[x^{n/2}(x^*_{2n}, y^*_{2n}) - x^{3n/2}(x^*_{2n}, y^*_{2n})]$$

where $x^m(x^*_n, y^*_n)$ is the x component of P^m; $P^m = T^m(x^*_n, y^*_n)$.

k	$n = 2^k$	δ_k	α_k	β_k
1	2	+8.302 027 955 067	−4.182 032 195 6760	
2	4	+8.658 536 606 728	−4.024 566 880 4350	+16.380 916 510 7200
3	8	+8.715 626 364 719	−4.020 265 962 4780	+16.374 036 107 6700
4	16	+8.720 272 377 757	−4.018 127 170 9880	+16.363 173 422 4000
5	32	+8.721 021 129 676	−4.018 114 128 1800	+16.364 325 866 8600
6	64	+8.721 157 158 380	−4.018 000 301 9390	+16.363 545 570 0000
7	128	+8.719 889 995 217	−4.018 574 541 8890	+16.366 227 999 6100
8	256	+8.720 551 061 678	−4.029 029 434 8210	+16.409 186 906 0200
9	512			

For example, the Feigenbaum sequence of the bifurcation parameters p_n and n-period fixed points, $P_n^{*\mathrm{DV}}$ for the De Vogelaere mapping can be obtained by

$$p_n = c_n \qquad P_n^{*\mathrm{DV}} = -P_n^{*\mathrm{HL}}/(1-c_n),$$

from the Feigenbaum sequence for Helleman's standard mapping, $\{c_n\}$, $\{P_n^{*\mathrm{HL}}\}$, where $P_n = (x_n^*, y_n^*)$.

Similarly the Feigenbaum sequence of Helleman's standard mapping can be obtained from that of Hénon's quadratic mapping by

$$c_n^{\pm} = 1 \pm (1+a_n)^{1/2} \tag{6}$$

and

$$P_n^{*\pm\mathrm{HL}} = -\tfrac{1}{2}a_n P_n^{*\mathrm{HN}} - \tfrac{1}{2}c_n^{\pm}. \tag{7}$$

In fact, a single Feigenbaum sequence studied by Bountis (1981) gives rise to double Feigenbaum sequences of Helleman (1980), which are called mirrors of each other by Helleman.

Table 2. Feigenbaum sequence of period-doubling bifurcations of Helleman's mapping (1) and (2) on the symmetry line $x = cy + y^2$ with accumulation point $c_c^- = -1.266311\ldots$. The 'mirror-Feigenbaum' sequence exists at $c_c^+ = 2 - c_c^- = 3.266311\ldots$. These are related to a_c of Hénon's by $c_c^{\pm} = 1 \pm (1+a_c)^{1/2}$. Although these sequences are calculated independently, they can be obtained from the sequences of table 1 via equations (6) and (7). (*a*) and (*b*) are similar to (*a*) and (*b*) of table 1 except that the *a*'s are replaced by the c^-'s.

(*a*).

k	$n=2^k$	c_k^-	$\phi_n'(a_k, y_n^*)$	x_n^*	y_n^*
0	1	−1.000 000 000 000	−2.000 000 000 000	+0.000 000 000 000	+0.000 000 000 000
1	2	−1.236 067 977 500	−2.000 000 000 000	−0.381 966 011 250	+0.618 033 988 750
2	4	−1.262 841 686 314	−2.000 000 000 001	−0.383 524 044 179	+0.508 261 446 465
3	8	−1.265 913 483 006	−2.000 000 001 600	−0.392 144 837 579	+0.540 818 806 676
4	16	−1.266 265 663 776	−2.000 000 002 570	−0.390 891 519 103	+0.533 304 660 957
5	32	−1.266 306 046 720	−2.000 000 011 200	−0.391 296 263 742	+0.535 242 414 909
6	64	−1.266 310 677 203	−2.000 000 412 300	−0.391 205 595 662	+0.534 767 904 765
7	128	−1.266 311 208 151	−1.999 987 191 000	−0.391 229 278 205	+0.534 886 886 943
8	256	−1.266 311 269 040	−1.999 579 680 000	−0.391 223 510 787	+0.534 857 380 530
9	512	−1.266 311 276 022	−2.015 570 330 000	−0.391 224 965 702	+0.534 864 762 325

(*b*).

k	$n=2^k$	δ_k	α_k	β_k
1	2	+8.817 156 380 537	−4.059 779 549 3300	
2	4	+8.715 976 836 530	−4.011 024 913 1630	+16.325 797 580 5000
3	8	+8.722 215 843 869	−4.018 714 464 8400	+16.367 717 051 7000
4	16	+8.721 027 619 977	−4.017 949 418 7710	+16.362 449 757 9200
5	32	+8.721 108 359 538	−4.018 093 106 4670	+16.364 241 413 0700
6	64	+8.721 161 017 651	−4.018 086 697 8960	+16.363 852 336 0500
7	128	+8.719 932 992 823	−4.018 381 465 7760	+16.365 138 385 4300
8	256	+8.720 853 623 604	−4.006 607 883 1480	+16.313 959 390 8600
9	512			

In tables 1 and 2, we present Feigenbaum sequences of the Hénon mapping (4) and Helleman's mapping (2) on the dominant symmetry line $x = h^{\mathrm{HN/HL}}(y)$ calculated according to the theory developed by the authors (see Lee and Choi 1983). They are indeed related to each other by the relations (6) and (7). In addition to the usual sequences leading to the universal scaling constant α, δ as well as the parameter sequence, the sequences leading to the second scaling constant β are formed. This is the sequence of the ratios of differences in the x components of n-period fixed points bifurcated off the symmetry line which is the $T^{n/4}$ image of the pair of fixed points bifurcated on the symmetry line as shown by the author (see Lee *et al* 1983). At low periods, the sequences appear quite different because the transformations (6) and (7) are not simple. Because of the extreme sensitivity of the stability criteria of high periods on the parameter (at period $n = 2^9 = 512$, one part in 10^{13} in the parameter value (keeping 10^{-13} for the accuracy of the fixed point) affects about one part in 100 in the value of $\phi'_n(y^*_n)$ which is related to the residue 'R' by $-2R$ (see Lee and Choi 1983)), the accuracy of the scaling constants δ_k, α_k, β_k at higher periods should not be taken too seriously.

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